The descriptive set theory of the Lebesgue density theorem

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The category algebra.

Work in some perfect Polish space, e.g. $\omega_2$. $\mathcal{B}$ is the collection of all sets with the property of Baire, $M_{GR}$ is the ideal of meager sets,

$$
\mathcal{B}/M_{GR} \cong \text{Bor}/M_{GR} = \text{Cat}
$$

$\text{Cat}$ is unique up-to isomorphism, i.e. it does not depend on the Polish space. The map

$$
\rho: \text{Cat} \to \text{RO}
$$

is a selector, and $\text{Cat}$ can be identified with the collection of all regular open sets.

$\text{Cat}$ is a Polish space.
The measure algebra.

\( \mu \) a continuous probability Borel measure on some perfect Polish space, e.g. the usual Lebesgue measure on \( \omega^2 \). \( \text{Meas} \) is the collection of all sets measurable sets, \( \text{Null} \) is the ideal of measure-zero sets,

\[
\text{Meas}/\text{Null} \cong \text{Bor}/\text{Null} = \text{Malg}
\]

\( \text{Malg} \) is unique up-to isomorphism, i.e. it does not depend on \( \mu \). \( \text{Malg} \) is a Polish space:

\[
\delta([A], [B]) = \mu (A \triangle B)
\]
The Lebesgue density theorem

Definition

\(x\) has density \(r \in [0; 1]\) in \(A \subseteq \omega^2\) if

\[
D_A(x) \overset{\text{def}}{=} \lim_{n \to \infty} \frac{\mu(A \cap N_x \upharpoonright n)}{\mu(N_x \upharpoonright n)} = r.
\]

Theorem (Lebesgue)

Let \(A \subseteq \omega^2\) be Lebesgue measurable. Then

\[
\Phi(A) = \{x \in \omega^2 \mid x \text{ has density } 1 \text{ in } A\}
\]

is Lebesgue measurable, and \(\mu(A \triangle \Phi(A)) = 0\).

In other words: \(D_A\) agrees with \(\chi_A\) almost everywhere.
The Lebesgue density theorem

If $\mu(A \triangle B) = 0$ then $\Phi(A) = \Phi(B)$, so

$$\Phi: \text{MALG} \rightarrow \text{MEAS}$$

is a selector. This is the analogue of $\rho: \text{CAT} \rightarrow \text{RO}$.

**Question**

What is the complexity of $\Phi(A)$?
Localization

Definition

The localization of $A$ at $s$ is

$$A_{[s]} = \left\{ x \in \omega^2 \mid s \upharpoonright x \in A \right\}$$

Thus $s \upharpoonright A_{[s]} = A \cap N_s$.

Trivial observation

$$\mu(A \triangle B) = 0 \iff \forall s \in \omega^2 \left( \mu(A_{[s]}) = \mu(B_{[s]}) \right)$$

Thus a measure class $[A]$ is completely determined by the map $s \mapsto \mu(A_{[s]})$. 
Density
A. Andretta, R. Camerlo

The motivation
Complete Boolean algebras
Lebesgue's theorem
The density topology

Results
Πᵢᵢ⁻¹-completeness
Wadge degrees
Dualistic sets

Comeagerness
Forcing
Φ is Borel

Would you like to see some proofs?
Πᵢᵢ⁻¹ completeness
Inside \( \Delta \)

Complexity of \( \Phi \)

Since

\[ x \in \Phi(A) \iff \forall k \exists n \forall m \geq n (\mu(A_{|x|m}) \geq 1 - 2^{-k-1}) \]

then

Proposition (Folklore)

For all measurable \( A \)

\[ \Phi(A) \in \Pi^0_3. \]

Question

Is \( \Pi^0_3 \) optimal?
The density topology

- \( A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B) \),
- \( \Phi(A \cap B) = \Phi(A) \cap \Phi(B) \),
- \( \bigcup_{i \in I} \Phi(A_i) \subseteq \Phi\left( \bigcup_{i \in I} A_i \right) \),
- if \( A \) is open, then \( A \subseteq \Phi(A) \).

Definition

\[ \mathcal{T} = \{ A \in \text{MEAS} \mid A \subseteq \Phi(A) \} \]

is the density topology. It is finer than the usual topology.
The density topology

Theorem (Scheinberg 1971, Oxtoby 1971)

\[ A = \Phi(A) \] if and only if \( A \) is open and regular in \( \mathcal{T} \).

\[ \Phi : \text{MALG} \rightarrow \text{RO}_{\mathcal{T}} \]

- \( \text{NULL} = \text{MGR}_{\mathcal{T}} \) (Oxtoby, 1971)
- \( \mathcal{T} \) is neither first countable, nor second countable, nor Lindelöf, nor separable.
- \( \mathcal{T} \) is Baire.
Recall that $\Phi(A)$ is always $\Pi^0_3$.

**Theorem**

There is an $A$ such that $\Phi(A)$ is complete $\Pi^0_3$.

Clearly

$$\text{Int}(A) \subseteq \Phi(A) \subseteq \text{Cl}(A).$$

and $A = \Phi(A)$ if $A$ is clopen.

**Question**

Can $\Phi(A)$ be something other than clopen or complete $\Pi^0_3$?

Yes!
Wadge degrees

Definition

A is Wadge reducible to B

\[ A \leq_W B \]

just in case \( A = f^{-1}(B) \) for some continuous \( f: \omega^2 \to \omega^2 \).

\[ A \equiv_W B \text{ iff } A \leq_W B \land B \leq_W A. \]

The equivalence classes \([A]_W\) are called Wadge degrees.

For \( d \subseteq \Pi^0_3 \) a Wadge degree, let

\[ \mathcal{W}_d = \{ [A] \mid \Phi(A) \in d \} \]
The sets $\mathcal{W}_d$ are non-empty, in fact are dense in the topological space $\text{MALG}$:

$$\forall \varepsilon \forall A \forall d \subseteq \Pi^0_3 \exists C \in \Pi^0_1 \exists U \in \Sigma^0_1$$

$$\left( \Phi(C) = \Phi(U) \in d \land \mu(A \triangle C) < \varepsilon \right).$$
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$\Pi^0_3$-completeness
Wadge degrees
Dualistic sets

Comeagerness
Forcing
$\hat{\Phi}$ is Borel

Would you like to see some proofs?
$\Pi^0_3$-completeness
Inside $\Delta^0_3$

Dualistic sets

Recall that $D_A(x) = 0, 1$ for almost all $x$.

Definition
A set $A$ is dualistic (or Manichæan) if $D_A(x) = 0, 1$ for all $x$. $M$ is the Boolean algebra of all dualistic sets.

Clearly being dualistic depends on the equivalence class of $A$, so

$$A \in M \iff \Phi(A) \in M.$$ 

Fact
$A = \Phi(A)$ is dualistic iff $A$ is $T$-clopen, i.e.,

$$M \cap \text{ran}(\Phi) = \Delta^0_1-T$$
Dualistic sets

Proposition

\[ \forall A \in \text{Meas} \ (A \in \mathcal{M} \Rightarrow \Phi(A) \in \Delta^0_2). \]

We can refine the Metric Approximation Theorem for \( \Delta^0_2 \) degrees:

\[ \forall \varepsilon > 0 \ \forall A \ \forall d \subseteq \Delta^0_2 \ \exists C \in \Pi^0_1 \ \exists U \in \Sigma^0_1 \ (\Phi(C) = \Phi(U) \in \mathcal{W}_d \cap \mathcal{M} \land \mu(A \triangle C) < \varepsilon) \]
A comeager set

Theorem

Let \( d = \Pi_3^0 \setminus \Delta_3^0 \) be the degree of the complete \( \Pi_3^0 \) sets. Then \( W_d \) is comeager in \( \mathcal{M}_{\text{ALG}} \).
Given any measurable $A$ there are $F \subseteq A \subseteq G$ with $F \in \Sigma_2^0$ and $G \in \Pi_2^0$ such that $\mu(A) = \mu(F) = \mu(G)$.

**Theorem**

$\{ [A] \mid [A] \cap \Delta_2^0 = \emptyset \}$ is comeager in $\text{MALG}$.  

Another comeager set

C’m on, we all knew that...
Dense sets in boolean algebras

By the Metric Approximation Theorem, the $\mathcal{W}_d$ are *topologically* dense in $\text{MALG}$. But $\text{MALG}$ is a Boolean algebra (i.e. a forcing notion) so there is a competing notion of *density*.

**Theorem**

Let $d = \Pi^0_3 \setminus \Delta^0_3$ be the degree of the complete $\Pi^0_3$ sets. If $\emptyset \neq A = \Phi(A)$ has empty interior, then $A \in d$.

*Therefore $\mathcal{W}_d$ contains a dense open set.*
Recall that $\Phi$ induces a map $\hat{\Phi} : MALG \to \Pi^0_3$,

$$\hat{\Phi}([A]) = \Phi(A).$$

Fix some standard coding $\pi : \omega^2 \to \Pi^0_3$

**Proposition**

$\hat{\Phi}$ is Borel, i.e. there is a Borel $\mathcal{F} : MALG \to \omega^2$ such that

$$\hat{\Phi}([A]) = \pi(\mathcal{F}([A])).$$
Sketch of the proof for $\Pi^0_3$ completeness

- $T$ a pruned tree such that $[T]$ has positive measure and empty interior. Thus $\neg[T] = \bigcup_n N_{t_n}$.
- $n < m \Rightarrow \text{lh}(t_n) < \text{lh}(t_m)$ and $\exists \omega \in (\text{lh}(t_n) + 1 < \text{lh}(t_{n+1}))$.
- For all $t \in T$ there is a shortest $s \supset t$ such that $s \notin T$. $s$ is the target of $t$.
- Let $\tau(t) = \text{lh}(\text{target of } t) - \text{lh}(t)$, $\tau : T \to \omega \setminus \{0\}$.
- For $x \in [T]$,
  $$ x \in \Phi([T]) \iff \lim_{n \to \infty} \tau(x \upharpoonright n) = \infty. $$
Sketch of the proof for $\Pi^0_3$ completeness, ctd.

The set

$$P = \{ z \in \omega \times \omega 2 \mid \forall m \forall n \ z(n, m) = 0 \}.$$ 

is complete $\Pi^0_3$.

Given $a : n \times n \to 2$ construct a node $\varphi(a) \in T$ so that

$$a \subset b \Rightarrow \varphi(a) \subset \varphi(b),$$

and

$$\omega \times \omega 2 \to [T], \quad z \mapsto \bigcup_n \varphi(z \upharpoonright n \times n)$$

witnesses $P \leq_w \Phi([T])$. 

Let \( a : (n + 1) \times (n + 1) \to 2 \). (Say \( n = 4 \))

**Case 1:**

\[
\begin{array}{cccc|c}
   a_{0,4} & a_{1,4} & a_{2,4} & a_{3,4} & 0 \\
   a_{0,3} & a_{1,3} & a_{2,3} & a_{3,3} & 0 \\
   a_{0,2} & a_{1,2} & a_{2,2} & a_{3,2} & 0 \\
   a_{0,1} & a_{1,1} & a_{2,1} & a_{3,1} & 0 \\
   a_{0,0} & a_{1,0} & a_{2,0} & a_{3,0} & 0 \\
\end{array}
\]

Then pick \( t \supset \varphi(a \upharpoonright n \times n) \) such that

\[
\tau(t) \geq \max \{ n + 1, \tau(\varphi(a \upharpoonright n \times n)) \}.
\]
Sketch of the proof for $\Pi^0_3$ completeness, ctd.

Let $a: (n + 1) \times (n + 1) \to 2$. (Say $n = 4$)

Case 2:

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Then pick $t \supset \varphi(a \upharpoonright n \times n)$ such that

$$\tau(t) = 3.$$
The Wadge hierarchy on $\omega_2$.

- A set $A$ (or degree) is self dual if $A \equiv_W \neg A$. Otherwise it is non-self-dual.
- Self-dual and non-self-dual pairs alternate.
- At all limit levels there is a non-self-dual pair.

limit level
Given $f : \omega \rightarrow \omega \setminus \{0\}$ and sets $A_0, A_1, \ldots$ consider the set

$$\text{Rake}^-(f; (A_n)_n)$$
How to construct larger degrees.

If $\exists \infty n (f(n) \geq 2)$ and the $A_n$s are $\mathcal{T}$-regular, i.e. $\Phi(A_n) = A_n$ then so is $\text{Rake}^- (f; (A_n)_n)$. Moreover

- if $A = A_0 = A_1 = \ldots$ are self-dual, then $\text{Rake}^- (f; (A_n)_n)$ is non-self-dual and immediately above $A$,
- if $A_0 <_W A_1 <_W A_2 <_W \ldots$ then $\text{Rake}^- (f; (A_n)_n)$ is non-self-dual and immediately above the $A_n$s.

Note that the rake $\text{Rake}^- (f; (A_n)_n)$ has no pole, i.e., $0^{(\infty)}$ does not belong to this set. In order to construct the dual degrees we need another kind of rake, a pole and densely packed tines.
How to construct larger degrees.

\[ \text{Rake}^+ (f; (A_n)_n) \]

\[ \omega_2 \]

\[ \omega_2 \]

\[ \omega_2 \]

\[ \omega_2 \]

\[ \omega_2 \]

\[ A_0 \]

\[ A_1 \]

\[ A_2 \]

\[ A_3 \]

\[ A_4 \]
How to construct larger degrees.

If \( \lim_{n} f(n) = \infty \) then and the \( A_n \)s are \( \mathcal{T} \)-regular, i.e. \( \Phi(A_n) = A_n \) then so is \( \text{Rake}^+(f; (A_n)_n) \). Moreover

\[
\text{Rake}^+(f; (A_n)_n) \equiv_{w} \neg \text{Rake}^-(f; (A_n)_n) .
\]

If \( A \) and \( B \) are \( \mathcal{T} \)-regular then so is

\[
A \oplus B = 0^\complement A \cup 1^\complement B .
\]

Arguing this way, we can climb up to \( \Delta^0_2 \).
Wadge defined two operations $A^\natural$ and $A^\flat$ on subsets of the *Baire space*

$$A^\natural = \left\{ s_0^+ s_1^+ s_2^+ \ldots s_n^+ x^+ \mid n \in \omega, s_i \in <\omega \omega, x \in A \right\}$$

$$A^\flat = A^\natural \cup \{ x \in \omega \omega \mid \exists \infty n \ (x(n) = 0) \}$$

where $s^+$ and $x^+$ are the sequences obtained from $s$ and $x$ by adding a 1 to all entries.

The idea is that $A^\natural$ is the union of $\omega$ many layers of the form

$$A^+ = \{ x^+ \mid x \in A \}$$
Jumping $\omega_1$ levels.

Theorem (Wadge)

If $A$ is self-dual, then $A^\ddagger$ and $A^\flat$ form a non-self-dual pair and

$$\|A^\ddagger\|_W = \|A^\flat\|_W = \|A\|_W \cdot \omega_1.$$ 

The operations $A^\ddagger$ and $A^\flat$ together with the (analsogs of) the Rake operations, are sufficient to construct sets of rank $< \omega_1^{\omega_1}$, i.e. the $\Delta^0_3$ sets.
Jumping $\omega_1$ levels.

An analogue of $A^+$. 

- $s \upharpoonright i = \bar{s} \upharpoonright ii$, for $s \in <\omega 2$.
- $\bar{x} = \bigcup_n x \upharpoonright n$, for $x \in \omega 2$.
- Replace $A$ with $\{\bar{x} \mid x \in A\}$, but...
- Does not work, since $\{\bar{x} \mid x \in \omega 2\}$ is of measure 0!
- The cure: enlarge $\{\bar{x} \mid x \in A\}$ like $\text{Rake}^-$ was enlarged to $\text{Rake}^+$. The resulting set is called $\text{Plus}(A)$.
- In fact we construct $\text{Plus}(A; r)$ (with $r \in (0; 1)$) so that $\mu \left( \text{Plus}(A; r)_{[\bar{s}]} \right) \geq r$ for all $s$.
- If $A$ is $\mathcal{T}$-regular (i.e., $A = \Phi(A)$), then so is $\text{Plus}(A; r)$. 
Jumping $\omega_1$ levels.

Construct $\text{Nat}(A)$ and $\text{Flat}(A)$: they are the analogs of $A^\sharp$ and $A^\flat$, and have rank $\|A\|_W \cdot \omega_1$.

Using the operations $\text{Nat}(A)$, $\text{Flat}(A)$, $\text{Rake}^{-} A$, $\text{Rake}^{+} A$, and $\oplus$ it is possible to construct a closed sets $C$ such that $\Phi(C)$ is of any given Wadge degree in $\Delta^0_3$. 
Nat(A)

Fix $0 < r < 1$. Nat(A) is composed of $\omega$-many layers:

\[
\begin{align*}
\text{Plus}(A; r) \\
\text{Plus}(A; r) \\
\text{Plus}(A; r)
\end{align*}
\]

- If $x$ settles inside a layer, then $x = s^\sim y$ and the density of $x$ in Nat(A) will be ‘similar’ to the density of $y$ in $A$.
- Every time we climb to a higher level, the density drops momentarily to $\leq 1/2$. So if $x$ climbs infinitely many layers, then $x$ will not have density 1 in Nat(A).
Fix $0 < r_0 < r_1 < r_2 < \cdots \to 1$.

Flat($A$) is the set is composed of $\omega$-many layers

\[
\vdots
\]

\[
\text{Plus}(A; r_2)
\]

\[
\text{Plus}(A; r_1)
\]

\[
\text{Plus}(A; r_0)
\]

- If $x$ settles inside a layer, then $x = s \upharpoonright y$ and the density of $x$ in Flat($A$) will be ‘similar’ to the density of $y$ in $A$.
- In the layer $n$, the density will always be $\geq r_n$. So if $x$ climbs infinitely many layers, then $x$ will have density 1 in Flat($A$).
Density

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