The descriptive set theory of the Lebesgue density theorem

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The category algebra.

Work in some perfect Polish space, e.g. ${}^{\omega}2$.
$\mathcal{B}$ is the collection of all sets with the property of Baire,
MGR is the ideal of meager sets,

$$\mathcal{B}/\mathrm{MGR} \cong \mathrm{BOR}/\mathrm{MGR} = \mathrm{CAT}$$

CAT is unique up-to isomorphism, i.e. it does not depend on the Polish space. The map

$$\rho\colon \mathrm{CAT} \to \mathrm{RO}$$

is a selector, and CAT can be identified with the collection of all regular open sets.
CAT is a Polish space.

The measure algebra.

μ a continuous probability Borel measure on some perfect Polish space, e.g. the usual Lebesgue measure on ${}^{\omega}2$.
MEAS is the collection of all sets measurable sets,
NULL is the ideal of measure-zero sets,

$$\mathrm{MEAS}/\mathrm{NULL} \cong \mathrm{BOR}/\mathrm{NULL} = \mathrm{MALG}$$

MALG is unique up-to isomorphism, i.e. it does not depend on μ. MALG is a Polish space:

$$\delta([A],[B]) = \mu\,(A \triangle B)$$

The Lebesgue density theorem

Definition

x has density $r \in [0;1]$ in $A \subseteq {}^{\omega}2$ if

$$D_A(x) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{\mu\left(A \cap \boldsymbol{N}_{x\restriction n}\right)}{\mu\left(\boldsymbol{N}_{x\restriction n}\right)} = r.$$

Theorem (Lebesgue)

Let $A \subseteq {}^{\omega}2$ be Lebesgue measurable. Then

$$\Phi(A) = \{x \in {}^{\omega}2 \mid x \text{ has density } 1 \text{ in } A\}$$

is Lebesgue measurable, and $\mu(A \mathbin{\triangle} \Phi(A)) = 0$.
In other words: D_A agrees with χ_A almost everywhere.

The Lebesgue density theorem

If $\mu(A \bigtriangleup B) = 0$ then $\Phi(A) = \Phi(B)$, so

$$\Phi \colon \text{MALG} \to \text{MEAS}$$

is a selector. This is the analogue of $\rho \colon \text{CAT} \to \text{RO}$.

Question

What is the complexity of $\Phi(A)$?

Localization

Definition

The localization of A at s is

$$A_{\lfloor s \rfloor} = \left\{ x \in {}^{\omega}2 \mid s^\frown x \in A \right\}$$

Thus $s^\frown A_{\lfloor s \rfloor} = A \cap \boldsymbol{N}_s$.

Trivial observation

$$\mu\left(A \triangle B\right) = 0 \Leftrightarrow \forall s \in {}^{<\omega}2 \left(\mu(A_{\lfloor s \rfloor}) = \mu(B_{\lfloor s \rfloor})\right)$$

Thus a measure class $[A]$ is completely determined by the map $s \mapsto \mu(A_{\lfloor s \rfloor})$

Complexity of Φ

Since

$$x \in \Phi(A) \Leftrightarrow \forall k \exists n \forall m \geq n \left(\mu\left(A_{\lfloor x \restriction m \rfloor}\right) \geq 1 - 2^{-k-1}\right)$$

then

Proposition (Folklore)

For all measurable A

$$\Phi(A) \in \mathbf{\Pi}^0_3.$$

Question

Is $\mathbf{\Pi}^0_3$ optimal?

The density topology

- $A \subseteq B \Rightarrow \Phi(A) \subseteq \Phi(B)$,
- $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$,
- $\bigcup_{i \in I} \Phi(A_i) \subseteq \Phi(\bigcup_{i \in I} A_i)$,
- if A is open, then $A \subseteq \Phi(A)$.

Definition

$$\mathcal{T} = \{A \in \textsc{Meas} \mid A \subseteq \Phi(A)\}$$

is the density topology. It is finer than the usual topology.

The density topology

Theorem (Scheinberg 1971, Oxtoby 1971)

$A = \Phi(A)$ if and only if A is open and regular in $\mathcal{T}$.

$$\Phi\colon \mathrm{MALG} \to \mathrm{RO}_{\mathcal{T}}$$

- $\mathrm{NULL} = \mathrm{MGR}_{\mathcal{T}}$ (Oxtoby, 1971)
- $\mathcal{T}$ is neither first countable, nor second countable, nor Lindelöf, nor separable.
- $\mathcal{T}$ is Baire.

$\mathbf{\Pi}^0_3$-completeness

Recall that $\Phi(A)$ is always $\mathbf{\Pi}^0_3$.

Theorem

There is an A such that $\Phi(A)$ is complete $\mathbf{\Pi}^0_3$

Clearly

$$\mathrm{Int}(A) \subseteq \Phi(A) \subseteq \mathrm{Cl}(A).$$

and $A = \Phi(A)$ if A is clopen.

Question

Can $\Phi(A)$ be something other than clopen or complete $\mathbf{\Pi}^0_3$?

Yes!

Wadge degrees

Definition

A is Wadge reducible to B

$$A \leq_{\mathrm{W}} B$$

just in case $A = f^{-1}(B)$ for some continuous $f \colon {}^{\omega}2 \to {}^{\omega}2$.

$A \equiv_{\mathrm{W}} B$ iff $A \leq_{\mathrm{W}} B \wedge B \leq_{\mathrm{W}} A$.
The equivalence classes $[A]_{\mathrm{W}}$ are called Wadge degrees.
For $\boldsymbol{d} \subseteq \mathbf{\Pi}^0_3$ a Wadge degree, let

$$\mathcal{W}_{\boldsymbol{d}} = \{[A] \mid \Phi(A) \in \boldsymbol{d}\}$$

Metric Approximation Theorem

The sets $\mathcal{W}_{\boldsymbol{d}}$ are non-empty, infact are dense in the topological space MALG:

$$\forall \varepsilon\, \forall A\, \forall \boldsymbol{d} \subseteq \mathbf{\Pi}^0_3\, \exists C \in \mathbf{\Pi}^0_1\, \exists U \in \mathbf{\Sigma}^0_1 \\ (\Phi(C) = \Phi(U) \in \boldsymbol{d} \wedge \mu(A \triangle C) < \varepsilon)\,.$$

Dualistic sets

Recall that $D_A(x) = 0, 1$ for *almost* all x.

Definition

A set A is dualistic (or Manichæan) if $D_A(x) = 0, 1$ for all x. $\mathcal{M}$ is the Boolean algebra of all dualistic sets.

Clearly being dualistic depends on the equivalence class of A, so

$$A \in \mathcal{M} \Leftrightarrow \Phi(A) \in \mathcal{M}.$$

Fact

$A = \Phi(A)$ is dualistic iff A is $\mathcal{T}$-clopen, i.e.,

$$\mathcal{M} \cap \operatorname{ran}(\Phi) = \mathbf{\Delta}^0_1\text{-}\mathcal{T}$$

Dualistic sets

Proposition

$\forall A \in \text{MEAS} \; \left(A \in \mathcal{M} \Rightarrow \Phi(A) \in \mathbf{\Delta}^0_2 \right)$.

We can refine the Metric Approximation Theorem for $\mathbf{\Delta}^0_2$ degrees:

$$\forall \varepsilon > 0 \, \forall A \, \forall \boldsymbol{d} \subseteq \mathbf{\Delta}^0_2 \, \exists C \in \mathbf{\Pi}^0_1 \, \exists U \in \mathbf{\Sigma}^0_1 \\ (\Phi(C) = \Phi(U) \in \mathcal{W}_{\boldsymbol{d}} \cap \mathcal{M} \;\wedge\; \mu(A \bigtriangleup C) < \varepsilon)$$

A comeager set

Theorem

Let $\boldsymbol{d} = \mathbf{\Pi}^0_3 \setminus \mathbf{\Delta}^0_3$ be the degree of the complete $\mathbf{\Pi}^0_3$ sets. Then $\mathcal{W}_{\boldsymbol{d}}$ is comeager in MALG.

Another comeager setC'm on, we all knew that. . .

Given any measurable A there are $F \subseteq A \subseteq G$ with $F \in \mathbf{\Sigma}^0_2$ and $G \in \mathbf{\Pi}^0_2$ such that $\mu(A) = \mu(F) = \mu(G)$.

Theorem

$\{[A] \mid [A] \cap \mathbf{\Delta}^0_2 = \emptyset\}$ *is comeager in* MALG.

Dense sets in boolean algebras

By the Metric Approximation Theorem, the $\mathcal{W}_{\boldsymbol{d}}$ are *topologically* dense in MALG. But MALG is a Boolean algebra (*i.e.* a forcing notion) so there is a competing notion of *density*.

Theorem

Let $\boldsymbol{d} = \mathbf{\Pi}^0_3 \setminus \mathbf{\Delta}^0_3$ *be the degree of the complete* $\mathbf{\Pi}^0_3$ *sets. If* $\emptyset \neq A = \Phi(A)$ *has empty interior, then* $A \in \boldsymbol{d}$.
Therefore $\mathcal{W}_{\boldsymbol{d}}$ *contains a dense open set.*

$\hat{\Phi}$ is Borel

Recall that Φ induces a map $\hat{\Phi} \colon \text{MALG} \to \mathbf{\Pi}^0_3$,

$$\hat{\Phi}([A]) = \Phi(A).$$

Fix some standard coding $\pi \colon {}^{\omega}2 \twoheadrightarrow \mathbf{\Pi}^0_3$

Proposition

$\hat{\Phi}$ is Borel, i.e. there is a Borel $\mathcal{F} \colon \text{MALG} \to {}^{\omega}2$ such that $\hat{\Phi}([A]) = \pi(\mathcal{F}([A]))$.

Sketch of the proof for Π^0_3 completeness

- T a pruned tree such that $[T]$ has positive measure and empty interior. Thus $\neg[T] = \bigcup_n \boldsymbol{N}_{t_n}$.
- $n < m \Rightarrow \mathrm{lh}(t_n) < \mathrm{lh}(t_m)$ and $\exists^\infty n\,(\mathrm{lh}(t_n) + 1 < \mathrm{lh}(t_{n+1}))$.
- For all $t \in T$ there is a shortest $s \supset t$ such that $s \notin T$. s is the target of t.
- Let $\tau(t) = \mathrm{lh}(\text{target of } t) - \mathrm{lh}(t)$, $\tau \colon T \to \omega \setminus \{0\}$.
- For $x \in [T]$,

$$x \in \Phi([T]) \Leftrightarrow \lim_{n\to\infty} \tau(x \restriction n) = \infty.$$

Sketch of the proof for $\mathbf{\Pi}^0_3$ completeness, ctd.

The set

$$P = \left\{ z \in {}^{\omega\times\omega}2 \mid \forall m \, \forall^\infty n \, z(n,m) = 0 \right\} .$$

is complete $\mathbf{\Pi}^0_3$.
Given $a \colon n \times n \to 2$ construct a node $\varphi(a) \in T$ so that

$$a \subset b \Rightarrow \varphi(a) \subset \varphi(b),$$

and

$${}^{\omega\times\omega}2 \to [T] , \quad z \mapsto \bigcup_n \varphi(z \restriction n \times n)$$

witnesses $P \leq_{\mathrm{W}} \Phi([T])$.

Sketch of the proof for Π^0_3 completeness, ctd.

Let $a\colon (n+1) \times (n+1) \to 2$. (Say $n = 4$)
Case 1:

$a_{0,4}$	$a_{1,4}$	$a_{2,4}$	$a_{3,4}$	0
$a_{0,3}$	$a_{1,3}$	$a_{2,3}$	$a_{3,3}$	0
$a_{0,2}$	$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	0
$a_{0,1}$	$a_{1,1}$	$a_{2,1}$	$a_{3,1}$	0
$a_{0,0}$	$a_{1,0}$	$a_{2,0}$	$a_{3,0}$	0

Then pick $t \supset \varphi(a \restriction n \times n)$ such that

$$\tau(t) \geq \max\{n+1, \tau(\varphi(a \restriction n \times n))\}.$$

Sketch of the proof for $\boldsymbol{\Pi}^0_3$ completeness, ctd.

Let $a\colon (n+1)\times(n+1)\to 2$. (Say $n=4$)

Case 2:

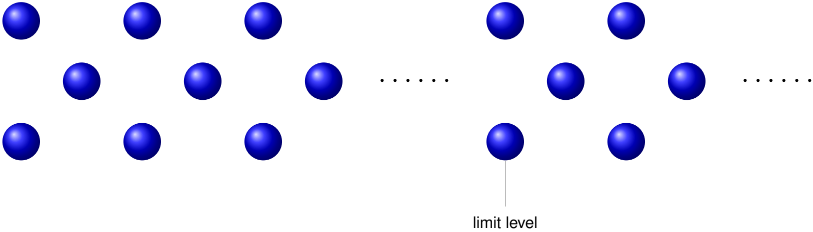
$a_{0,4}$	$a_{1,4}$	$a_{2,4}$	$a_{3,4}$	$\boldsymbol{a_{4,4}}$
$a_{0,3}$	$a_{1,3}$	$a_{2,3}$	$a_{3,3}$	$\boldsymbol{a_{4,3}}$
$a_{0,2}$	$a_{1,2}$	$a_{2,2}$	$a_{3,2}$	$\boldsymbol{a_{4,2}}$
$a_{0,1}$	$a_{1,1}$	$a_{2,1}$	$a_{3,1}$	$\boldsymbol{0}$
$a_{0,0}$	$a_{1,0}$	$a_{2,0}$	$a_{3,0}$	$\boldsymbol{0}$

Then pick $t\supset\varphi(a\restriction n\times n)$ such that

$$\tau(t)=3.$$

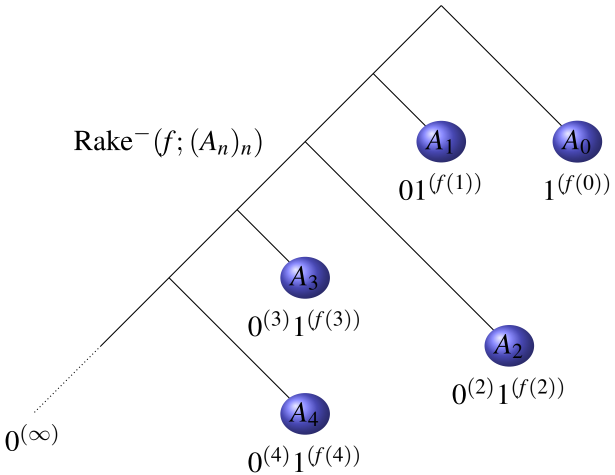
The Wadge hierarchy on ${}^{\omega}2$.

- A set A (or degree) is self dual if $A \equiv_{\mathrm{W}} \neg A$. Otherwise it is non-self-dual.
- Self-dual and non-self-dual pairs alternate.
- At all limit levels there is a non-self-dual pair.



How to construct larger degrees.

Given $f\colon \omega \to \omega \setminus \{0\}$ and sets $A_0, A_1, \ldots$ consider the set



How to construct larger degrees.

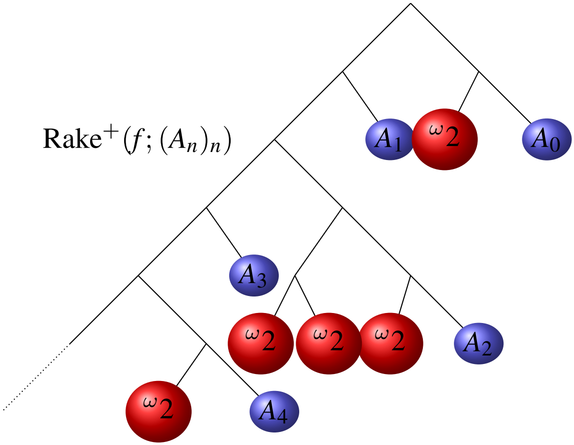
If $\exists^\infty n\,(f(n) \geq 2)$ and the A_ns are $\mathcal{T}$-regular, i.e. $\Phi(A_n) = A_n$ then so is $\mathrm{Rake}^-(f; (A_n)_n)$. Moreover

- if $A = A_0 = A_1 = \ldots$ are self-dual, then $\mathrm{Rake}^-(f; (A_n)_n)$ is non-self-dual and immediately above A,
- if $A_0 <_{\mathrm{W}} A_1 <_{\mathrm{W}} A_2 <_{\mathrm{W}} \ldots$ then $\mathrm{Rake}^-(f; (A_n)_n)$ is non-self-dual and immediately above the A_ns.

Note that the rake $\mathrm{Rake}^-(f; (A_n)_n)$ has no pole, i.e., $0^{(\infty)}$ does not belong to this set. In order to construct the dual degrees we need another kind of rake, a pole and densely packed tines.

How to construct larger degrees.

$\mathrm{Rake}^+(f; (A_n)_n)$



How to construct larger degrees.

If $\lim_n f(n) = \infty$ then and the A_ns are $\mathcal{T}$-regular, i.e. $\Phi(A_n) = A_n$ then so is $\mathrm{Rake}^+(f; (A_n)_n)$. Moreover

$$\mathrm{Rake}^+(f; (A_n)_n) \equiv_{\mathrm{W}} \neg \mathrm{Rake}^-(f; (A_n)_n).$$

If A and B are $\mathcal{T}$-regular then so is

$$A \oplus B = 0^\frown A \cup 1^\frown B.$$

Arguing this way, we can climb up to $\boldsymbol{\Delta}^0_2$.

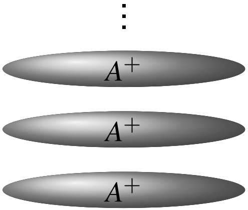
Jumping ω_1 levels.

Wadge defined two operations $A^\natural$ and $A^\flat$ on subsets of the *Baire space*

$$A^\natural = \left\{ s_0^+ {}^\frown 0 {}^\frown s_1^+ {}^\frown 0 {}^\frown \dots {}^\frown s_n^+ {}^\frown 0 {}^\frown x^+ \mid n \in \omega, s_i \in {}^{<\omega}\omega, x \in A \right\}$$

$$A^\flat = A^\natural \cup \{x \in {}^\omega\omega \mid \exists^\infty n \; (x(n) = 0)\}$$

where s^+ and x^+ are the sequences obtained from s and x by adding a 1 to all entries.
The idea is that $A^\natural$ is the union of ω many layers of the form



$$A^+ = \{x^+ \mid x \in A\}$$

Jumping ω_1 levels.

Theorem (Wadge)

If A is self-dual, then $A^\natural$ and $A^\flat$ form a non-self-dual pair and

$$\left\|A^\natural\right\|_{\mathrm{W}} = \left\|A^\flat\right\|_{\mathrm{W}} = \|A\|_{\mathrm{W}} \cdot \omega_1 .$$

The operations $A^\natural$ and $A^\flat$ together with the (analogs of) the Rake operations, are sufficient to construct sets of rank $< \omega_1^{\omega_1}$, i.e. the $\mathbf{\Delta}^0_3$ sets.

Jumping ω_1 levels.

An analogue of A^+.

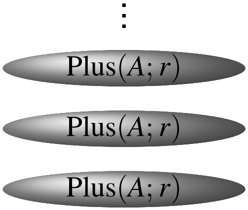
- $\overline{s^\frown i} = \bar{s}^\frown ii$, for $s \in {}^{<\omega}2$.
- $\bar{x} = \bigcup_n \overline{x \restriction n}$, for $x \in {}^{\omega}2$.
- Replace A with $\{\bar{x} \mid x \in A\}$, but...
- Does not work, since $\{\bar{x} \mid x \in {}^{\omega}2\}$ is of measure 0!
- The cure: enlarge $\{\bar{x} \mid x \in A\}$ like Rake^- was enlarged to Rake^+. The resulting set is called $\mathrm{Plus}(A)$.
- In fact we construct $\mathrm{Plus}(A; r)$ (with $r \in (0; 1)$) so that $\mu\left(\mathrm{Plus}(A; r)_{\lfloor \bar{s} \rfloor}\right) \geq r$ for all s.
- If A is $\mathcal{T}$-regular (i.e., $A = \Phi(A)$), then so is $\mathrm{Plus}(A; r)$.

Jumping ω_1 levels.

Construct $\mathrm{Nat}(A)$ and $\mathrm{Flat}(A)$: they are the analogs of $A^\natural$ and $A^\flat$, and have rank $\|A\|_{\mathrm{W}} \cdot \omega_1$.
Using the operations $\mathrm{Nat}(A)$, $\mathrm{Flat}(A)$, $\mathrm{Rake}^- A$, $\mathrm{Rake}^+ A$, and $\oplus$ it is possible to construct a closed sets$C$ such that $\Phi(C)$ is of any given Wadge degree in $\mathbf{\Delta}^0_3$.

Nat(A)

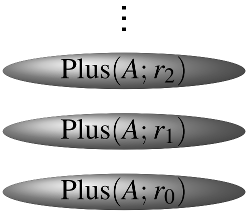
Fix $0 < r < 1$. Nat(A) is composed of ω-many layers



- If x settles inside a layer, then $x = s^\frown \bar{y}$ and the density of x in Nat(A) will be 'similar' to the density of y in A.
- Every time we climb to a higher level, the density drops momentarily to $\leq 1/2$. So if x climbs infinitely many layers, then x will not have density 1 in Nat(A).

$\mathrm{Flat}(A)$

Fix $0 < r_0 < r_1 < r_2 < \cdots \to 1$.
$\mathrm{Flat}(A)$ is the set is composed of ω-many layers



- If x settles inside a layer, then $x = s^\frown \bar{y}$ and the density of x in $\mathrm{Flat}(A)$ will be 'similar' to the density of y in A.
- In the layer n, the density will always be $\geq r_n$. So if x climbs infinitely many layers, then x will have density 1 in $\mathrm{Flat}(A)$.


That's all Folks!